DOCUMENT RESUME

ED 255 541 TM 850 113

AUTHOR Winsberg, Suzanne; And Others
TITLE Fitting Item Characteristic Curves with Spline Functions. [Program Statistics Research Technical Report Series].
INSTITUTION Educational Testing Service, Princeton, NJ. Program Statistics Research Project.
SPONS AGENCY Air Force Human Resources Lab., Brooks AFB, Texas.; Natural Sciences and Engineering Research Council, Ottawa (Ontario).
REPORT NO ETS-RR-84-40; PSR-TR-84-52
PUB DATE Dec 84
CONTRACT F41689-82-C-0020
GRANT A4035
NOTE 25p.
PUB TYPE Reports - Research/Technical (143)

EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS Difficulty Level; Estimation (Mathematics); Goodness of Fit; *Item Analysis; *Latent Trait Theory; Mathematical Models; *Maximum Likelihood Statistics; Test Items
IDENTIFIERS *Item Characteristic Function; *Monotone Splines

ABSTRACT

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Fitting Item Characteristic Curves With Spline Functions

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Program Statistics Research
Technical Report No. 84-52

Research Report No. 84-40

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December 1984

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This research was supported by operating grant No. A4035 from the Natural Sciences and Engineering Research Council of Canada. S. Winsberg, Principal Investigator.

This research was supported in part by contract No. F41689-82-C-0020 from the Air Force Human Resources Laboratory to the Educational Testing Service. H. Wainer, Principal Investigator.

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Abstract

In most item response theory models a particular mathematical form is assumed for all item characteristic curves, e.g., a logistic function. It could be desirable, however, to estimate the shape of the item characteristic curves without prior restrictive assumptions about its mathematical form. We have developed a practical method of estimating the shape of item response functions leading to smooth flexible functions and numerically stable calculations. In our approach the probability of success of an item is represented as a monotone function of the ability evaluated by the item. The function is represented as a monotone spline. We use the technique developed by Bock and Aitkin (1981) for marginal maximum likelihood estimation of the item parameters. Examples of real and artificial data are presented.

Keywords: Item response theory, item characteristic curves, monotone splines, marginal maximum likelihood estimation.

Introduction

Item response theory starts with a mathematical statement as to how an examinee's response to an item depends on his level of ability. This relationship is given by the item characteristic curve (ICC). An ICC relates the probability of success on a test item to the ability evaluated by the set of items which contain it. In most item response theory (IRT) models, a particular mathematical form is assumed for all ICCs, for example, normal ogive, step function, polynomial, logistic.

The usual rationale given for accepting any of these particular models is its fit to observed data. Therefore, it would be desirable to have a method of estimating the shape of an item's characteristic curve without prior restrictive assumptions about its mathematical form. One such method using polynomials of high degree has been developed by Lord (1970). The overall objective of this paper is to develop explicitly a well-conditioned parameterization of the ICCs which provides smooth, flexible functions and numerically stable calculations. In our approach the probability of success of an item is represented as a monotone function of the ability evaluated by the item. The function we choose for this representation is a monotone spline.

Although conditional maximum likelihood estimation is available for a limited number of models (the one parameter logistic) for which a sufficient statistic for ability exists, (Anderson 1973; 1980), in general one must integrate over the ability distribution and estimate the structural parameters of the IRT model by maximum likelihood in the marginal distribution (MML).

Recent advances (Bock and Aitkin, 1981) in item parameter estimation for IRT have paved the way for economically estimating the parameters in general item response models such as the one we are proposing. Much of the previous work done in MML estimation of item parameters has been what we shall refer to as (p,p).* That is, it is characterized by models with a parametric

*We are grateful to Paul Holland for pointing out this characterization.

representation of the item response curve and a parametric representation of the ability distribution. These parametric representations have limited flexibility. Bock and Aitkin (1981) and Bock and Lieberman (1970) use the normal cdf and Thissen (1982) the one parameter logistic family to represent the item characteristic curve. In each case the ability distribution was fixed, generally, as N (0,1). Bock and Aitkin (1981) and Thissen (1982) have concentrated on finding techniques for practical and efficient estimation of the parameters, a step crucial to the practical development of models incorporating a more flexible representation of the ICC. Holland and Cressie (1983), on the other hand, have taken what may be referred to as a (p,n) approach, using a parametric representation of the item response function (one parameter logistic) but a nonparametric representation of the ability distribution.

This paper complements these by considering a method approaching (n,p). That is, the ability distribution is fixed as N(0,1), but the item response function is allowed great flexibility. The approach we have taken cannot be said to be exactly (n,p) in character (because of the parametric structure of the family of splines used), but because of the generality of the spline function it approaches (n,p).

We have represented the ICC as a monotone spline function, previously described by Winsberg and Ramsay (1980; 1981; 1983) and by Ramsay (1982). They may be used to represent an item response function as they have the characteristics of a cumulative distribution function while offering the advantage of great flexibility of shape as well as a fairly parsimonious representation in the parameter space. Of great importance, while the effect of a parameter change in most functions is global, with a spline function it is more local. Thus inaccuracies of estimation at one end of a function's domain might cause reverberations throughout that domain in, say, a logistic representation, it would be kept in the region of the inaccuracy with a spline.

We shall proceed by first describing the IRT model, including a brief description of monotone splines, followed by an outline of the EM-like technique first described by Bock and Aitkin (1981) and then by Thissen (1982), which we have used to estimate the item parameters. Finally we present some examples.

Monotone Spline Representation of Item Characteristic Curves

Since there is no rationale for restricting the form of the ICC to a particular mathematical form, it seems sensible to represent this function in a more general way as a monotone function of ability.

Two approaches to the problem of representing monotone functions to be estimated have predominated in the past. In the first (Box and Cox, 1964; Andrews, et al, 1971), the use of power transformations, is often not sufficiently flexible. In the other approach, isotonic regression is applied resulting in a step function (Kruskal, 1965; Kruskal and Shepard, 1974) which is neither smooth, nor expressible in terms of a limited number of parameters. Monotone splines as an alternative to these approaches are a powerful family of smooth transformations amenable to statistical treatment which are extremely flexible and represented by a limited number of parameters. Moreover, splines are attractive as they are easy to compute and lead to numerically stable calculations. The generation and application of monotone splines are discussed by Winsberg and Ramsay (1980, 1981, 1983); an extensive treatment of splines is provided by de Boor (1978).

The ICC is a function of θ_i a set of ability parameters for person i and a set of item parameters η_j for item j. Let $x_{ij} = 1$ if subject i responds correctly to item j and $x_{ij} = 0$ otherwise. Denote the probability of success of subject i for item j as $P_{ij} = P(x_{ij} = 1|\theta_i, \eta_j)$. Commonly used models specify a logistic function for P_{ij} based on one, two or three item parameters (Lord, 1980).

One possible ICC is to represent the ability of subject i as a single parameter θ_i and P_{ij} as a monotone spline function of θ_i.

Spline functions are piecewise functions with a countable and usually finite number of junction points. In general only a subset of the possible splines need be considered. We shall restrict our attention to piecewise polynomials. To represent these piecewise polynomials one may fix in advance the maximal degree k-1, or order k, of the polynomial, the closed interval of interest (r,s) on which the splines are represented, and the number and location of a strictly increasing sequence

of junction points, the first of which is located at r and the last at s. We shall further restrict our discussion to the special case in which the spline function has maximal continuity at each interior junction point. In this case the function and its first k-2 derivatives are continuous at each junction point. The dimensionality of the space of spline functions, m, then becomes equal to the order plus the number of interior junction points.

It is useful to define a suitable basis for splines; that is, a sequence of splines such that any spline can be represented as a linear combination of members of this sequence. The M-splines provide a suitable basis for spline functions because they have the following properties: they are easy to compute; they lead to numerically stable calculations; and they may be used to generate monotone functions. To compute the basis functions for a given interval and set of junction points, one specifies a knot sequence $\{t_q\}$ q=1,...m+k, such that $t_{k+1} \cdots t_m$ lie at the $m-k$ interior junction points. The first and final k knots must lie outside the region of interest (r,s) but apart from this restriction can be chosen arbitrarily since they only affect the behavior of the function outside of (r,s). It is usually convenient to set $t_1 = \ldots = t_k = r$ and $t_{m+1} = \cdots = t_{m+k} = s$. An M-spline is zero everywhere except within the interval (t_q, t_{q+k}) where it is positive.* We have found that the case k=2 is especially useful. Let w be defined by $t_w < \theta < t_{w+1}$ for a specified argument θ. The explicit expressions for the two order 2 M-splines which are non-zero at θ are:

$$M_{w-1,\,2}(\theta) = (\Delta_{w1}\,\Delta_{w-1,2})^{-1}\,(t_{w+1} - \theta), \qquad (1)$$
$$M_{w,2}(\theta) = (\Delta_{w1}\,\Delta_{w2})^{-1}\,(\theta - t_w)$$

where

$$\Delta_{qk} = \frac{(t_{q+k} - t_q)}{k}$$

In applications requiring the generation and fitting of monotone spline functions, it is natural to construct as a basis the family of I-splines defined as

$$I_{qk}(\theta) := \int_r^{\theta} M_{qk}(u)\,du. \qquad (2)$$

* B-splines are related to M-splines by a normalization factor.

Thus an I-spline of order k is a piecewise polynomial of degree k. In the useful special case of k=2.

$$I_{w-1,2}(\theta) = 1-(\Delta_{w1}\,\Delta_{w-1,2})^{-1}(t_{w+1}-\theta)^2/2 \qquad (3)$$
$$I_{w2}(\theta) = (\Delta_{w1}\,\Delta_{w2})^{-1}(\theta-t_w)^2/2$$

From the non-negativity of M-splines, we have the monotonicity of the I-splines. Moreover, the M-splines are normalized such that $I_{qk}(s) = 1$. So one may represent a monotone spline function by $F(\theta) = \sum n_q\, I_{qk}(\theta)\, \eta_q > 0, q = 1,\ldots,m$.

Figure 1a shows a typical I-spline basis with k=2 and interior knots and Figure 1b shows a particular linear combination of this basis.

Our experience leads us to favor the use of k=2 with a few interior knots. Interior knots should be placed where one expects to find the greatest nonlinearity as well as including some data points between consecutive knots. We have found that this choice is both parsimonious and retains sufficient flexibility. Therefore we may represent

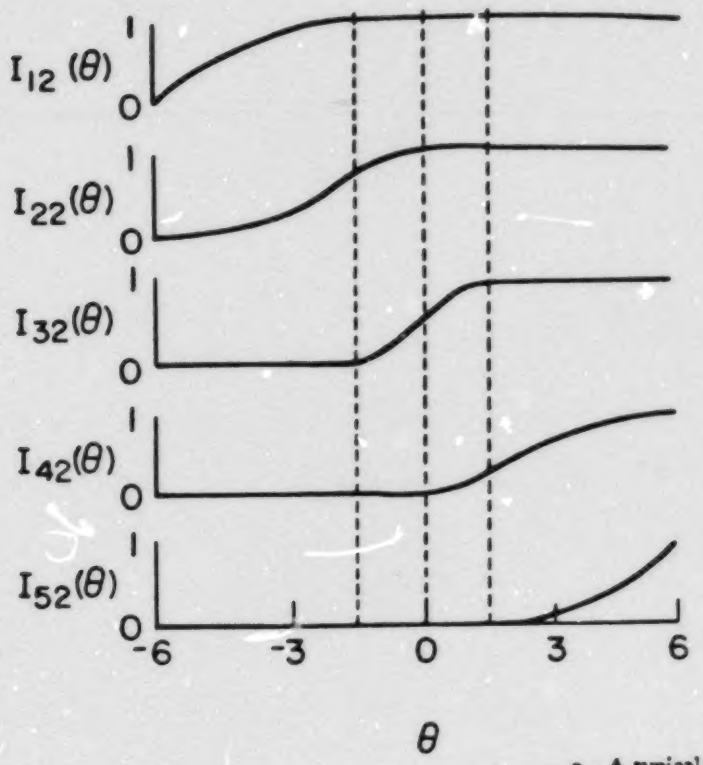
$$P_{ij} = P(x_{ij} = 1|\theta_i) \text{ as } P_{ij} = \sum_{q=1}^{m} \eta_{qj}\, I_{qj}(\theta_i) + \eta_{0j}\,. \qquad (4)$$

P_{ij} will be positive provided that $\eta_q \geq 0$, q=0,..., m and $p_{ij} \leq 1$ provided that $\sum_{q=0}^{m} \eta_{qj} = 1$. The item parameter n_{0j} is optional and may be thought of as a guessing parameter. We choose to define the splines on the interval (r,s,), $P_{ij}(r) = \eta_{0j}$ and $P_{ij}(s) = 1$. Strictly speaking the interval of interest is $(-\infty,\infty)$, but for practical purposes one may decide to define the splines from (-r,r), where r=5, thus limiting the interval of interest to one over which the ability distribution is not effectively zero.

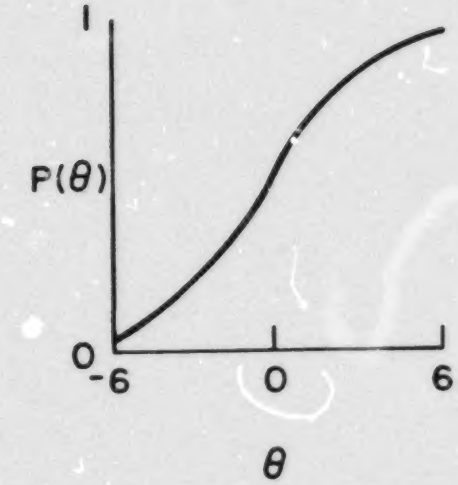
Parameter Estimation

The rationale for the Bock-Aitkin algorithm is presented in their paper (1981). The solution following the Bock-Aitkin algorithm requires data consisting of the number of responses N_g for the g^{th} observed response pattern, where there are at most 2^n response patterns for n items. The

Figure 1



a - A typical I-spline basis with k=2.



b - A particular linear combination of the basis shown in Figure 1a.

Bock-Aitkin algorithm has a two-step cycle. In the first step (the E-step) an initial table of expected response counts is formed conditionally on the data and the current item parameter estimates. In the second step (the M step) the item parameters are estimated conditionally on the current values of the expected response counts. The two steps are repeated cyclically until the process stabilizes.

The first step (the E-step) consists of computing

$$r^{*}_{lj\,1} = \textit{expected number of correct responses to item } j \textit{ at } \theta = U_l$$

and

$$r^{*}_{lj\,0} = \textit{expected number of incorrect responses to item } j \textit{ at } \theta = U_l,$$

where $U_l (l = 1,\ldots,v)$ are arbitrarily chosen values of θ. The U_l act as quadrature points in numerical integration and may be conveniently chosen to be equally spaced across the domain for which Φ has appreciable density. If $\Phi(\theta)$ is Gaussian, the domain -5 to 5 is adequate. So the first step of the algorithm consists of computing

$$r^{*}_{lj\,1} = \sum_{g}^{2n} N_g \, x_{gj} \, L_g (U_l) \tag{5}$$

$$r^{*}_{lj\,0} = \sum_{g}^{2n} N_g \, (1 - x_{gj}) \, L_g (U_l) \tag{6}$$

where

$$L_g (U_l) = \frac{\left\{ \prod_{j}^{n} P_{lj} (U_l)^{x_{gj}} \left[1 - P_{lj} (l) \right]^{1 - x_{gj}} \right\} \Phi (U_l)}{\sum_{l} \left\{ \prod_{j}^{n} P_{lj} (U_l)^{x_{gj}} \left[1 - P_{lj} (U_l) \right]^{1 - x_{gj}} \right\} \Phi (U_l)}$$

$$P_{lj} (U_l) = \sum_{q=1}^{m} \eta_{qj} \, I_{qk} (U_l) + \eta_{oj} \quad ,$$

and the g^{th} response pattern is denoted $x_g = (x_{g1}, \ldots, x_{gn})$.

The second step (the M-step) of the algorithm is the estimation of the item parameters by maximum likelihood using the artificial data. The log likelihood is given by

$$\log L (r^{*}_{lj\,1}, r^{*}_{lj\,0} \mid \eta_1, \ldots, \eta_j) = \tag{7}$$

$$- \sum_l \sum_j r^*_{lj\,1} \log P_{lj}(U_l) + r^*_{lj\,0} \log (1 - P_{lj}(U_l)) \quad .$$

The de.ivatives with respect to the item parameters are given by

$$D_{\eta_{qj}} \log L^* = \sum_l I_{qk}(U_l)\,[G]$$

$$q = 1,\ldots,m$$
$$j = 1,\ldots,n$$

$$D_{\eta_{oj}} \log L^* = \sum_l [G] \qquad (8)$$

where

$$G = \left\{ \frac{r^*_{lj\,1}}{P_{lj}(U_l)} - \frac{r^*_{lj\,0}}{1-P_{lj}(U_l)} \right\}$$

$$D^2_{\eta_{qj}\,\eta_{q'j}} \log L^* = \sum_l \left[I_{qk}(U_l)\, I_{q'k}(U_l)\,[H] \right.$$

$$q = 1,\ldots,m$$
$$q' = 1,\ldots,m$$
$$j = 1,\ldots,n$$

$$D^2_{\eta_{oj}\,\eta_{oj}} \log L^* = \sum_l [H] \qquad (9)$$

$$j = 1,\ldots,n$$

$$D^2_{\eta_{qj}\,\eta_{oj}} \log L^* = D^2_{\eta_{oj'}\,\eta_{qj}} \log L^* = \sum_l I_{qk}(U_l)\,[H]$$

$$q = 1,\ldots,m$$
$$j = 1,\ldots,n$$

$$D^2_{\eta_{q'j'}\,\eta_{qj}} \log L^* = 0$$

$$j \neq j'$$
$$q = 0,\ldots,m$$
$$q' = 0,\ldots,n$$

where

$$H = - \left\{ \frac{r^*_{lj\,1}}{P^2_{lj}(U_l)} + \frac{r^*_{lj\,0}}{1-P_{lj}(U_l)^2} \right\}$$

This last equation implies that one may optimize with respect to the item parameters or each item in turn since the optimum conditions for the other items will not be affected. Thus one may use a Newton-Raphson type of procedure involving the inversion of a small (m+1) x (m+1)

matrix even when the number of items is large.

One way to solve the resultant nonlinear programming problem with linear constraints is the penalty or barrier function approach. However, an attractive alternative to this approach is a technique in which at each iteration the parameters are linearly transformed and then a subset of them consisting of active constraints is held fixed. A constraint is considered active when a coefficient η_{qj} has become zero and the corresponding gradient element is negative. We must also take account of the equality constraint $1' \eta_j = 1$. An algorithm for using active and equality constraints is presented in Winsberg and Ramsay (1983).

In short the Bock-Aitkin algorithm consists of:

Fix initial values for item parameters

Step 1: Compute $r^*_{ij\,1}$ and $r^*_{ij\,0}$ as in (5) and (6).

Step 2: Estimate each of the η_j in turn by optimizing (7) subject to the constraints. If any of the η_{qj} change, repeat Step 1.

Each step is simple, however, the process converges slowly generally requiring many cycles.

Examples

To illustrate the model, we applied the Bock-Aitkin algorithm for MML estimation to two sets of data.

Example 1

The LSAT Section 6 data (5 items). The guessing term was fixed at zero. Quadratic I-solines were used, with two knots yielding three free parameters per variable. The same knot locations (0.0,1.1) were used for each item. The exterior knots were placed at -6 and 6. The ability distribution was taken to be N(0,1). Twenty-one quadrature points were used (evenly spaced between -5 and 5) because we wished to include a few quadrature points between each pair of knots. Observed and expected frequencies are shown in Table 1. Two item characteristic curves are shown in Figure 2 along with those obtained by Thissen (1982) for the one parameter logistic

Table 1

Observed and Expected Frequencies for LSAT data*

g	$\hat{N}_g$	N_g	X_g
Group	*Expected*	*Observed*	*Score Pattern*
32	293	298	11111
28	180	173	11011
20	75	81	10011
24	83	80	10111
30	62	61	11101
26	55	56	11001
18	34	29	10001
31	28	28	11110
22	27	28	10101
27	25	21	11010
12	13	16	01011
25	11	16	11000
23	12	15	10110
16	15	15	01111
19	15	14	10010
29	9	11	11100
4	8	11	00011
17	9	10	10000
10	6	8	01001
2	5	6	00001
8	6	4	00111
21	6	3	10100
14	5	3	01101
1	2	3	00000
7	1	3	00110
3	2	2	00010
15	2	2	01110
6	3	1	00101
9	2	1	01000
5	1	1	00100
11	3	0	01010
13	1	0	01100

$\chi^2 = 19$

* k=2, 2 interior knots at 0.0, and 1.1.

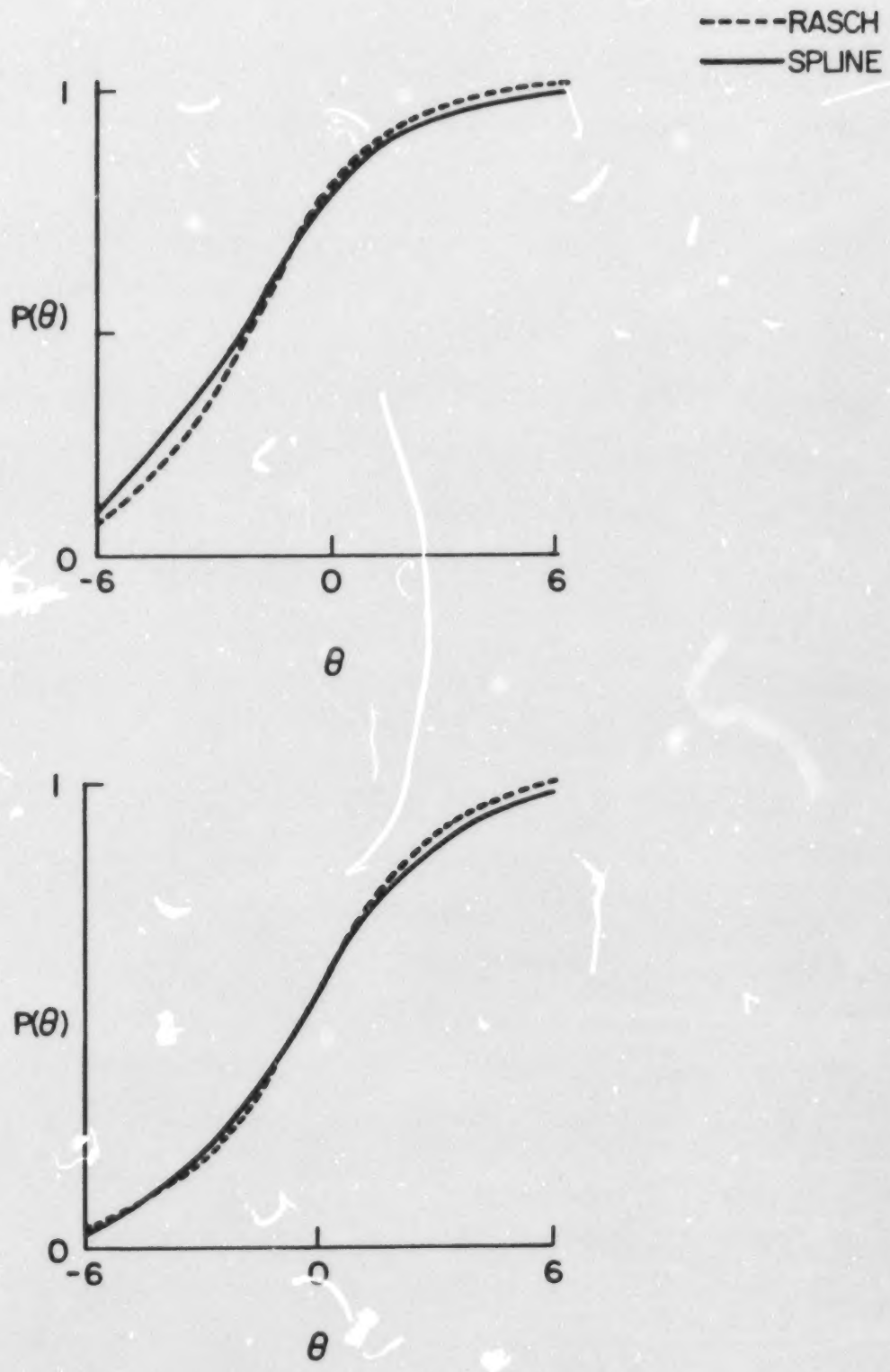


Figure 2

Two ICC's from the LSAT section 6 data (5 items). Both the spline model and Rasch model solutions are shown.

model. Solutions were also obtained for the cases of one and three interior knots. There was a significant improvement in the likelihood ratio χ^2 from 33 to 19 (with 5 degrees of freedom), in going from one to two interior knots (increasing from two to three free parameters per variable) but no significant improvement in going from two to three interior knots ($\chi^2_{(5)} = 1.3$) thus increasing from three to four free parameters per variable. Adding a guessing parameter did not significantly improve fit. The solution was fairly robust to knot placement; some results are presented in Table 2. The results indicate that it is the number of parameters (or interior knots) which determine the quality of fit rather than the location of the knots for these data. Placing knots in the region of greatest nonlinearity did not improve fit much.

In general, if one had many items it would not be practical to use different knot locations for each item. However, it might be feasible to have, say, one set of knot locations for easy items, another set for medium items and a third set for hard items.

Example 2

This example is artificial and demonstrates the algorithm's ability to recover a set of item characteristic curves. Trace curves for five Rasch items and an item with a plateau were used to generate true proportions for each of the 64 response patterns. We chose five Rasch items to demonstrate that a spline model would fit a Rasch characteristic curve reasonably well (obviously the Rasch model is best at fitting Rasch items). We added a plateau to demonstrate the usefulness of the spline model as a means of estimating the form of the ICC. A plateau might occur in an ICC if over a range of ability a particular distractor no longer was functioning as such and the remaining distractors did not become less appealing with increasing ability in that range. Or it might occur if there was a hole in the ability distribution.

Assuming that the observations in the 2^n categories are multinominally distributed we generated five sets of data for N=5000. These artificial data were analyzed using the spline model with integrated order two splines and three interior knots. The true and recovered ICC's are shown in figure 3. Observed and expected frequencies are shown for a typical case in Table 3.

Table 2

Chi Square Values Obtained for LSAT data for different Spline Models

Number of interior knots/item	Location of interior knots	χ^2
1	0.0 for all items	33
2	0.0 and 1.1 for all items	19
2	0.5 1.5 for all items	18
2	0.5 1.75 for all items	18
2	-1.0, 1.0 item 1 0.5, 1.5 item 2 0.5, 1.5 item 3 0.0, 1.5 item 4 -1.0, 1.0 item 5	18
3	-1.5 0 1.5 all items	17

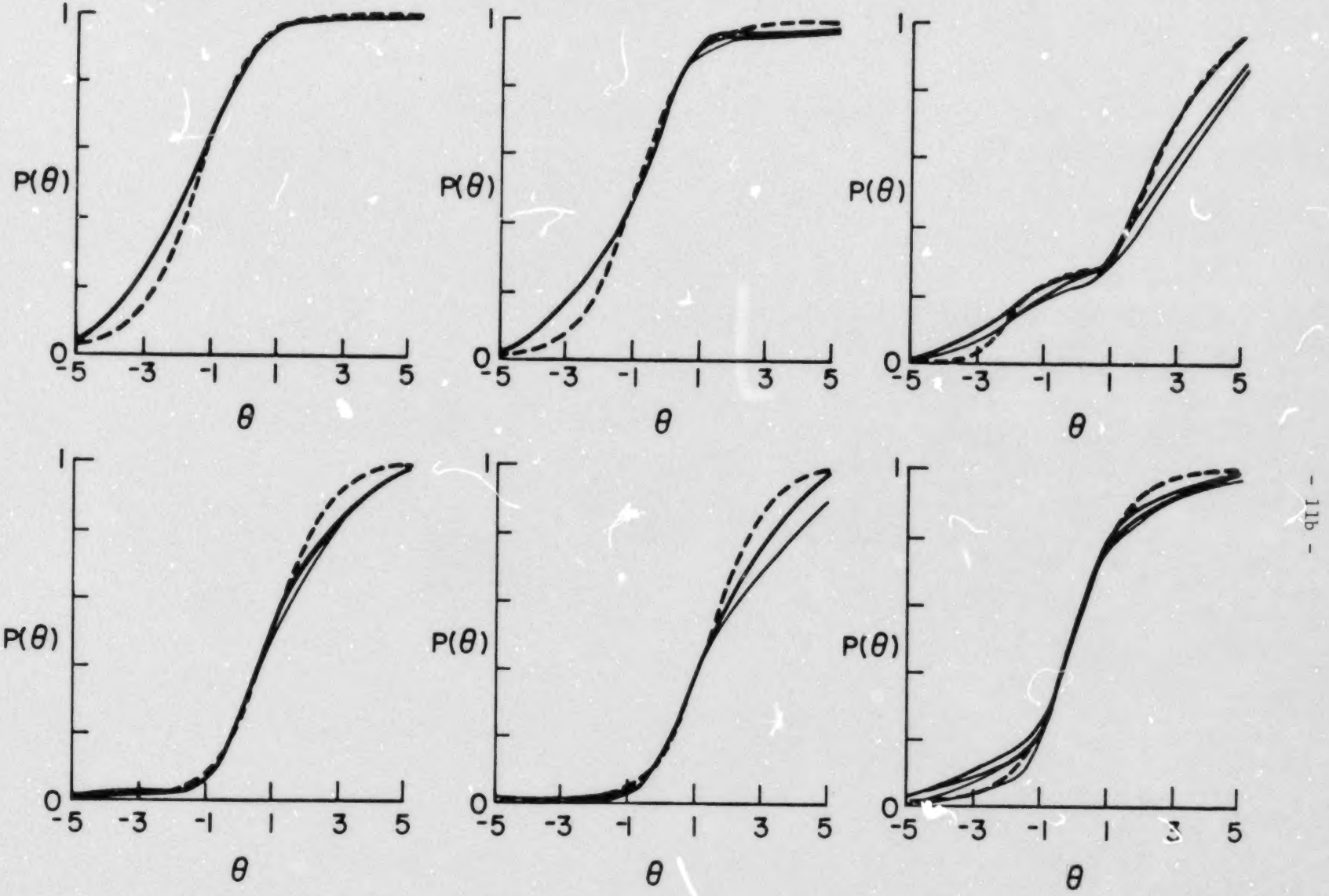


Figure 3

The true and recovered ICC's for the five artifical data sets containing six items that were generated for N = 5000.

The ICC's are well recovered except at the tails where there are few observations. The lack of fit in the tails is exacerbated if N is reduced to 1000 or less. These results indicate that if N is not sufficiently large a variety of solutions are equally valid and will yield similar χ^2 values.

Conclusion

The spline model appears to be sufficiently flexible to represent a wide variety of forms for the ICC. If used in an exploratory manner to obtain an estimate of the form of the ICC, it would protect against the possibility of model misspecification that might occur when using other ICC models. It might also be used as a a bench mark against which to test other models. It would be imprudent to try to estimate the form of the ICC for small samples.

Table 3

Observed and Expected Frequencies
for one Example of the Artificial Data

i	$\hat{N}_i$	N_i	X_i	$\frac{(N_i-\hat{N}_i)^2}{\hat{N}_i}$	$\sum_1^i \frac{(N_i-\hat{N}_i)^2}{N_i}$
47	641	652	110000	.189	.189
52	602	604	110001	.007	.196
46	401	390	100000	.302	.498
41	327	324	110101	.028	.526
1	222	242	000000	1.800	2.326
64	225	232	111001	.218	2.544
16	216	202	010000	.907	3.451
49	186	198	111000	.774	4.225
58	188	188	110011	0.00	4.225
50	172	181	110100	.471	4.696
32	182	169	100001	.929	5.627
42	165	169	110111	.091	5.718
43	159	135	111101	3.623	9.341
44	112	124	111111	1.286	10.627
17	98	101	010011	.092	10.719
48	97	95	101000	.041	10.760
63	95	92	111011	.095	10.855
51	90	78	110010	1.600	11.456
55	63	65	111100	.063	11.518
2	62	63	000010	.016	11.534
35	52	57	101011	.481	12.015
54	52	52	101001	0.00	12.015
24	52	45	110000	.942	12.957
57	51	44	110110	1.255	14.212
36	47	39	100101	1.361	15.573
9	45	39	001000	.800	16.373
21	28	33	010101	.893	17.266
56	34	32	111010	.118	17.384
20	28	31	010100	.321	17.705
62	26	31	111110	.962	18.667
25	29	30	011001	.034	19.301
34	24	29	100011	1.042	20.343
33	22	29	100010	2.228	22.571
60	16	23	101101	3.063	25.634
10	15	21	001001	2.400	28.034
19	14	13	010011	.071	28.105
61	12	13	100111	.083	28.188
5	16	12	000100	1.00	29.188
18	12	12	010010	0.00	29.188
29	10	12	011101	.400	29.588
23	8	9	010111	.125	29.713
28	8	9	011100	.125	29.838
38	7	9	101010	.571	30.409
37	7	9	100110	.571	30.980
13	4	9	001100	6.25	37.230
39	9	8	101011	.111	37.341
4	4	8	000011	4.00	41.341
53	15	7	101100	4.267	45.608
6	8	5	000101	1.125	46.733
40	6	5	101111	.167	46.900
22	4	4	010110	0.00	46.900
3	5	3	000010	.8	47.7
27	5	3	011011	.8	48.5
31	4	3	100100	.250	48.75
59	3	2	101110	.333	49.083
14	2	2	010000	0.000	49.083
26	4	1	011101	2.25	51.333
30	2	1	100011	.50	51.833
11	1	1	001100	0.00	51.833
12	1	1	001101	0.00	51.833
7	1	0	000111	1.00	52.833
8	1	0	001000	1.00	53.833
15	0	0	010001	0.00	53.833
45	0	0	001110	0.00	53.833

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